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JAN 79 D B DEBRA, J V BREAKWELL, M KUROSAKI F19628-78-C-0038

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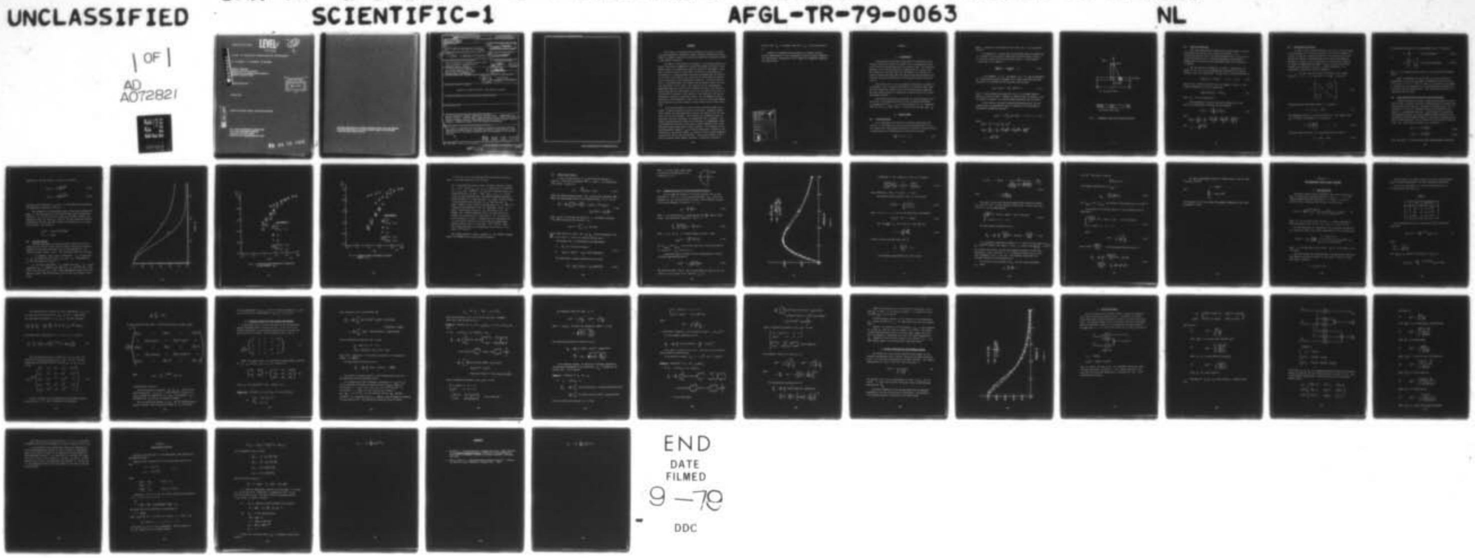

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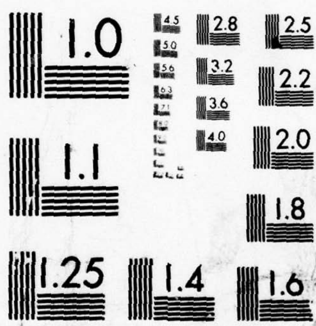

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STUDY TO DEVELOP GRADIOMETER TECHNIQUES

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report is concerned with the problem of real-time filtering of gravity-gradiometer readings onboard a uniformly horizontally moving vehicle to produce estimates of the gravity deflection and gravity anomaly at the vehicle's position.

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FOREWORD

This report is concerned with the problem of real-time filtering of gravity-gradiometer readings on board a uniformly horizontally moving vehicle to produce estimates of the gravity deflection and gravity anomaly at the vehicle's position.

The report is divided into two chapters. In the first chapter the gravity variations are modeled as arising from random variations in mass density along a line immediately below the vehicle at some depth. Using first a discrete version of this model in which the state consists of the density at a moving sequence of position extending from far behind to far in front of the vehicle, the covariance of the steady-state optimal filter is obtained by the Chandrasekhar algorithm. To confirm these results, a continuous model is introduced and the optimal filter is obtainable from Wiener filter theory. The solutions of the corresponding Wiener-Kopf equations for the filter transfer functions take relatively simple forms when the measurement consists of the spatial derivative, in the direction of motion, of the quantity to the estimated, and when the measurement accuracy is sufficiently high. The corresponding transfer functions in this "asymptotic" limit are of first or second order depending on whether the gravity anomaly or gravity deflection is being estimated.

In the second chapter, a more realistic continuous model, due to Heller, is used to describe the gravity variations. Asymptotic Wiener filter results for high measurement accuracy are found to take the same form as in the first chapter. These results are extended to cases in which more than one measurement is incorporated into an estimate. The results concerning gravity deflection estimation are unambiguous: cross-track deflection needs only first-order transfer-functions, but in-line deflection needs second-order transfer functions. The inclusion of the gravity-gradient component Γ_{xx} (x being the direction of motion) in the estimation of the gravity anomaly g_z does not lead to simple asymptotic forms for the filter transfer functions. The results obtained in Ch. I with the discrete model suggest, moreover, a substantial increase in accuracy when Γ_{xx} is included, along with Γ_{xz}, in the estimation of g_z.

Chapter II concludes with the outline of a method, based on rational approximation to the transcendental spectral density components, for constructing a steady-state filter without the asymptotic approximation used earlier.

CHAPTER I

A. INTRODUCTION

The disturbance mass model which causes the deflection of the vertical and gravity anomaly is assumed to be a one-dimensional horizontal line mass distribution below the vehicle's path. The stochastic property of the line mass density is considered to be a white noise. The intensity of the white noise and the depth of the line mass are chosen to produce the same root mean square values and correlation distance for the gravity deflection as the measured values on the earth surface.

The optimal filter for the gradiometer measurement is sought by two methods. First, we convert the model into a discrete model; then find the steady-state optimal filter by the Chandrasekhar algorithm. The other method consists of finding the stationary filter by solving a Wiener-Hopf equation.

The results obtained by both methods are in good agreement with each other. Of noteworthy interest is that the double measurement, Γ_{zx} together with $\frac{1}{2}(\Gamma_{xx} - \Gamma_{zz})$ yields information on g_x comparable with that from Γ_{xx} alone, and on g much better than that from Γ_{xz}.

B. DISCRETE MODEL

B-1 System Equations

In a coordinate frame fixed to the vehicle which moves with velocity $\bar{v}$ with respect to the earth, the mass distribution of the earth seems to be time-varying and the rate of change of the mass density is described by the following partial differential equation:

$$\frac{\partial\rho}{\partial\rho} + \bar{v} \cdot \nabla\rho = 0 , \qquad (1\text{-}1)$$

where ρ denotes the mass density of the earth, and ∇ is the gradient operator.

For simplicity, we assume that the disturbance mass is concentrated on a straight line with finite length below the vehicle path, see Fig. I-1, when the vehicle travels with constant speed v toward negative x direction, (1.1) reduces to

$$\frac{\partial\rho(t,x)}{\partial t} - v\,\frac{\partial\rho(t,x)}{\partial x} = 0 \,. \tag{1.2}$$

At the boundary $x = -\ell$, the density $\rho(t, -\ell)$ may be considered as a random process which is assumed to be a white noise with zero mean because we are interested in only deviation from the mean:

$$E\left\{\rho(t,-\ell)\rho(t',-\ell)\right\} = \frac{q}{V}\,\delta(t-t') \tag{1.3}$$

where E denotes expectation operator, and q is the power spectral density of the white noise. Though (1.2) does not have any process noise, the boundary condition (1.3) always brings uncertainty into the system.

On the vehicle, the gravity and gravity gradient due to the disturbance mass are expressed in terms of integrals of the mass density multiplied by weighting functions:

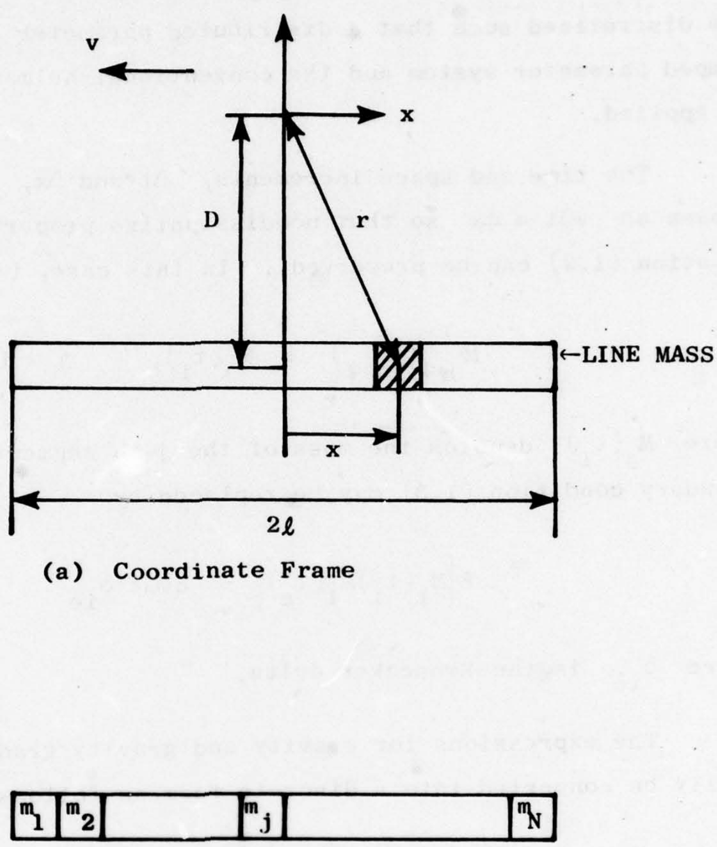
$$Z_k(t) = \gamma \int_{-\ell}^{\ell} w_k \rho(t,x)\,dx, \quad k = 1,2,\ldots,5 \tag{1.4}$$

where

$$[Z_k] = [g_x,\ g,\ \Gamma_{xx},\ \Gamma_{zz},\ \Gamma_{xz}]^T$$

$$[w_k] = \left[\frac{x}{r^3},\ -\frac{D}{r^3},\ \frac{-r^2+3x^2}{r^5},\ \frac{-r^2+3D^2}{r^5},\ \frac{-3Dx}{r^5}\right]^T$$

$$r = \sqrt{x^2+D^2}\,.$$



(a) Coordinate Frame

(b) Discrete Mass Model

FIG. 1 COORDINATE FRAME AND DISCRETE MASS MODEL

B-2 Discrete Formulation

The formal application of Kalman filter theory leads to a partial differential equation for the states error covariance . To avoid this difficulty, here, the system equations (1.2), (1.3) and (1.4) are discretized such that a distributed parameter system becomes a lumped parameter system and the conventional Kalman filter theory can be applied.

The time and space increments, Δt and Δx, respectively, are chosen as $v\Delta t = \Delta x$ so that nondissipative property of the original equation (1.2) can be preserved). In this case, (1-2) may be written as

$$M_{j+1}(t_{i+1}) = M_j(t_i), \qquad j = 1, \ldots, N-1 \tag{1.5}$$

where $M_j(t_i)$ denotes the mass of the j-th segment at time t_i. The boundary condition (1.3) may be replaced by

$$E\left\{M_1(t_i)M_1(t_e)\right\} = qv\Delta t\ \delta_{ie} \tag{1.6}$$

where δ_{ie} is the Kronecker delta.

The expressions for gravity and gravity gradient (1.4) may easily be converted into a discrete form as follows:

$$Z_k(t_i) = \gamma \sum_{j=1}^{N} W_{kj}\ M_j(t_i) \tag{1.7}$$

where

$$[W_{kj}] = \left[\frac{x_j}{r_j^3}, \frac{D}{r_j^3}, \frac{-r_j^2 + 3x_j^2}{r_j^5}, \frac{-r_j^2 + 3D^2}{r_j^5}, \frac{-3Dx_j}{r_j^5}\right]^T$$

and $r_j = \sqrt{x_j^2 + D^2}$.

B-3 Chandrasekhar Algorithm

In order to make a good approximation, we have to take the length of the line mass, 2ℓ, as large as possible, and the space increment Δx, as small as possible. When the parameters of the system are not time-varying but constant, the Chandrasekhar-type algorithm developed by Kailath et al. [Ref. 1] has the possibility of substantially reducing the amount of computation and computer capacity required below those necessary with the Riccati equation. A brief explanation of the Chandrasekhar algorithm is given in Appendix 1.

our case, the state vector is the mass of each segment $\{M_j(t_i)\}^T$, $j = 1, \ldots, N$, and matrix F in App. 1 is given by

$$F = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & \ddots & & 0 & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & & 1 & 0 \end{bmatrix} \tag{1.8}$$

The process noise distribution matrix G is given by

$$G = [1, 0, \ldots, 0]^T . \tag{1.9}$$

The measurement matrix H is obtained from (1.7). For example, when Γ_{xz} is used as a measurement, H is given by

$$H = \left\{ \gamma \frac{-3Dx_j}{r_j^5} \right\}, \qquad j = 1, \ldots, N . \tag{1.10}$$

The power spectral density Q of the process noise is given by

$$Q = qv\Delta t = q\Delta x . \tag{1.11}$$

The power spectral density of the measurement noise R is given by

$$R = \frac{r_c}{\Delta t} \qquad \text{for one measurement} \tag{1.12a}$$

or

$$R = \begin{bmatrix} \frac{r_c}{\Delta t} & 0 \\ 0 & \frac{r_c}{\Delta t} \end{bmatrix} \qquad \text{for a double measurement} \tag{1-12b}$$

where r_c is the power spectral density of the continuous measurement noise.

Since the dimension of process noise is only one and that of measurement is one or two, the number of computations for one step is on the order of $N^2 \times 2$ or $N^2 \times 3$. Hence, as the number of the segment N becomes large, the superiority of the Chandrasekhar algorithm over the Riccati equation becomes clearer. The latter needs an order of N^3 computations for one step.

B-4 Power Spectral Density and Auto-correlation Function

Before we proceed to numerical computation discussed in the previous section, we mention the gravity field produced by an infinite line mass with white noise spectrum. Extending the integral limits in (1.4) to infinity, (1.4) may be regarded as convolution integrals of mass density ρ and weighting functions. Since the Fourier transform of the weighting functions are given by modified Bessel functions of the second kind, the power spectral density of the gravity and gravity gradient are easily obtained. For the deflection of the vertical along the track, g_x, and the gravity anomaly g, we find that

$$\Phi_{g_x}(\omega) = 4\gamma^2 q \omega^2 K_0^2(D\omega) \tag{1.13a}$$

$$\Phi_g(\omega) = 4\gamma^2 q \omega^2 K_1^2(D\omega) \tag{1.13b}$$

where K_0 and K_1 are zero and first-order modified Bessel functions, respectively. The rms values of g_x and g are given by

$$(g_x)_{rms} = \frac{\Upsilon}{2} \sqrt{\pi q / 2D^3} \tag{1.14a}$$

$$(g)_{rms} = \frac{\Upsilon}{2} \sqrt{3\pi q / 2D^3} \quad . \tag{1.14b}$$

Auto-correlation functions of g_x and g are obtainable not analytically but numerically, and shown in Fig. I-2.

Two parameters to be determined, namely, white noise intensity and depth D, are chosen such that the resulting rms value and correlation distance of g_x are the sea-level values $g_0 \times 8$ arcsec, and 20 n. mi., respectively [Ref. 2], where $g_0 = 9.8$ m/s^2 and the correlation distance is defined as the shift distance at which the ACF drops to $1/e$ of that for zero shift. The result is:

$$\begin{cases} \gamma^2 q = 1.65 \times 10^{-8} \text{ km}^5/\text{sec}^4 \\ D = 36 \text{ km} \end{cases} \quad .$$

B-5 Numerical Results

We have investigated the estimation error covariance of the deflection of the vertical and the gravity anomaly, when we used different components of the gravity gradient tensor as the measurements. The results are shown in Figs. 3 and 4. The conclusions drawn are as follows.

1. As expected, among "single" measurements, Γ_{xx} is preferable to Γ_{zx} for estimating its integral g_x while Γ_{zx} is preferable to Γ_{xx} for estimating its integral g.

2. The double measurement, Γ_{zx} together with $\frac{1}{2}(\Gamma_{xx} - \Gamma_{zz})$ yields information on g_x comparable with that from Γ_{xx} alone and on g much better than that from Γ_{xz} alone. These measurements consist of the outputs of a single rotating gradiometer with its spin axis aligned to the g-axis.

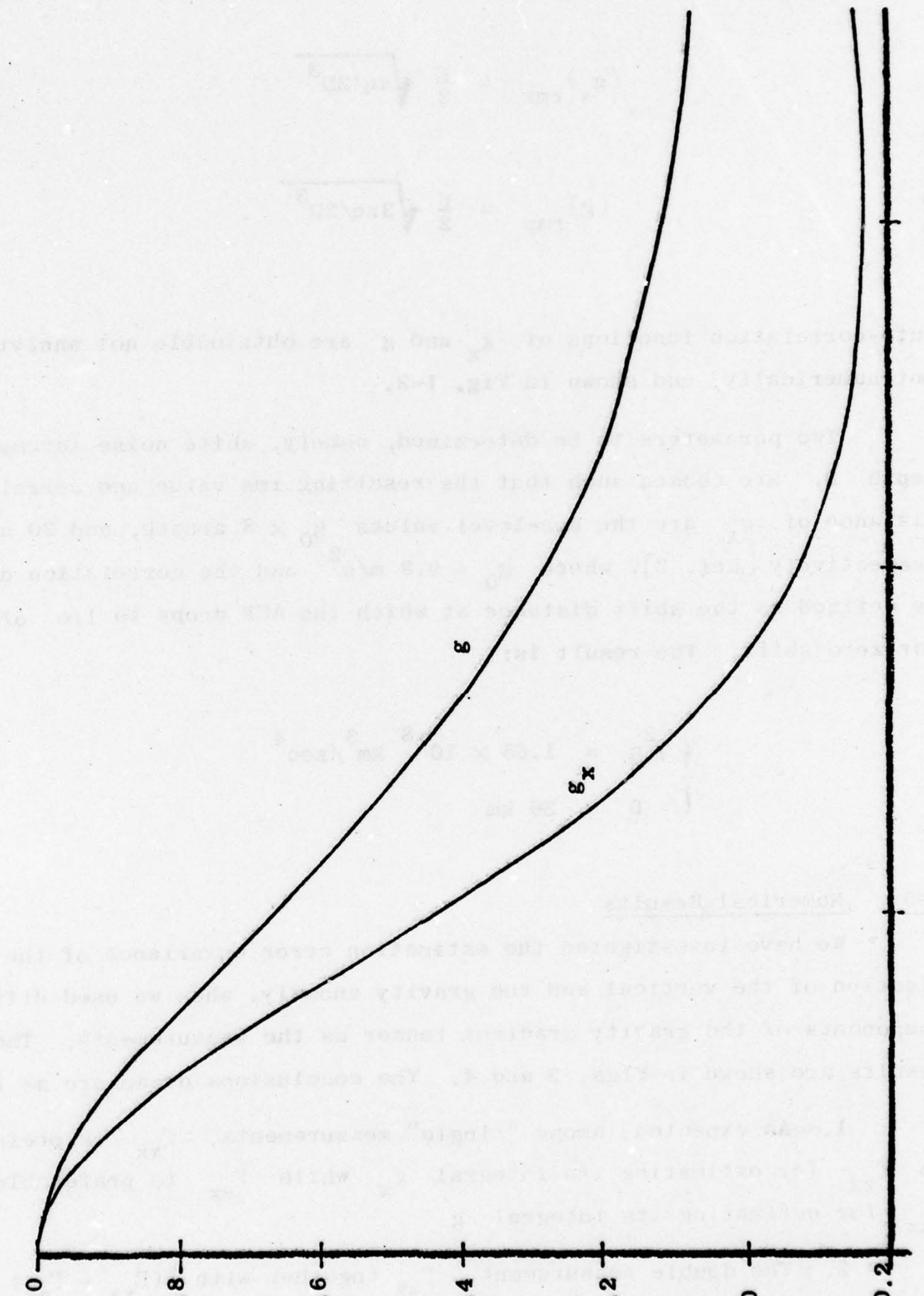


FIGURE I-2

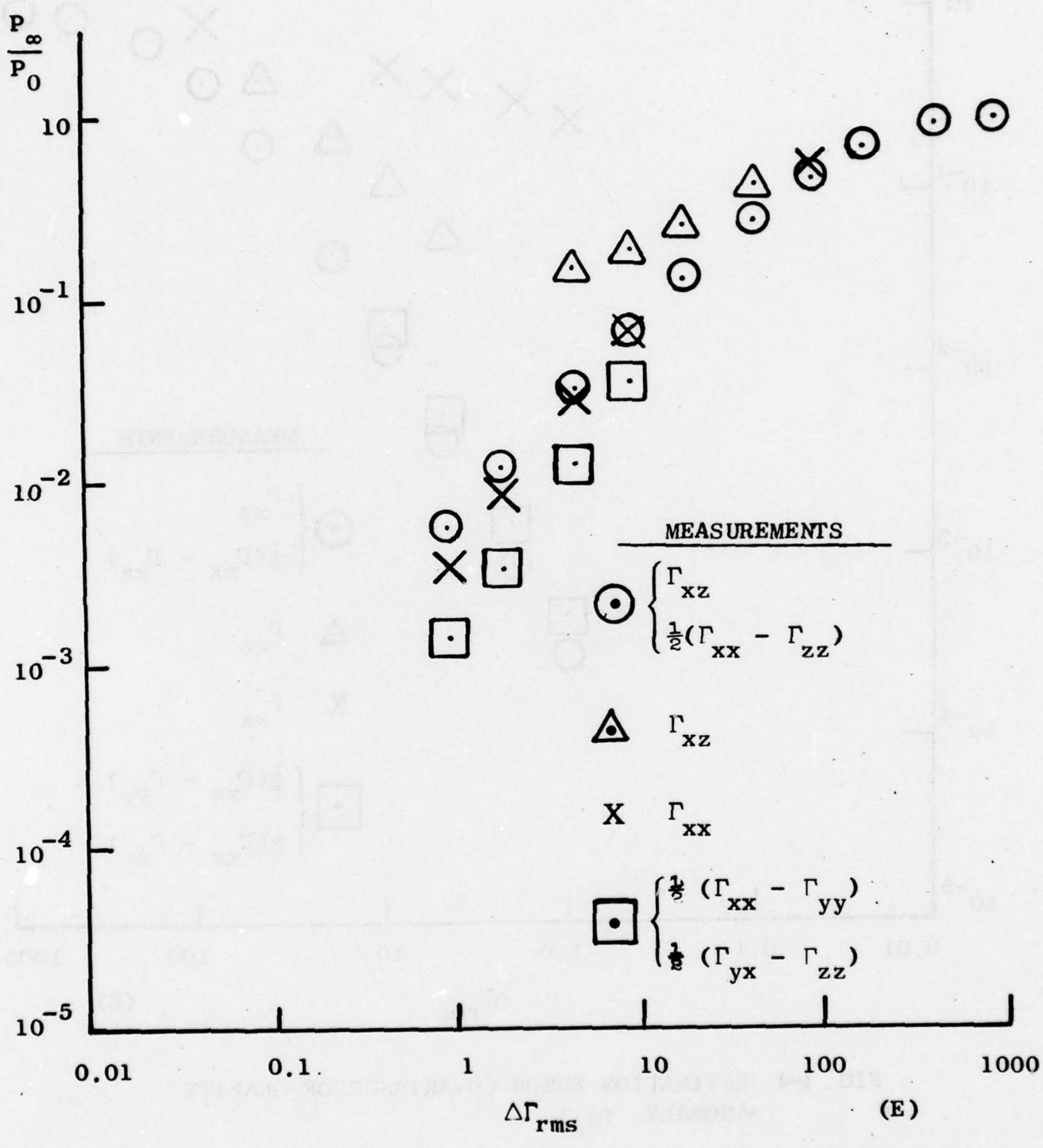


FIG. I-3 ESTIMATION ERROR COVARIANCE OF DEFLECTION OF THE VERTICAL (g_x)

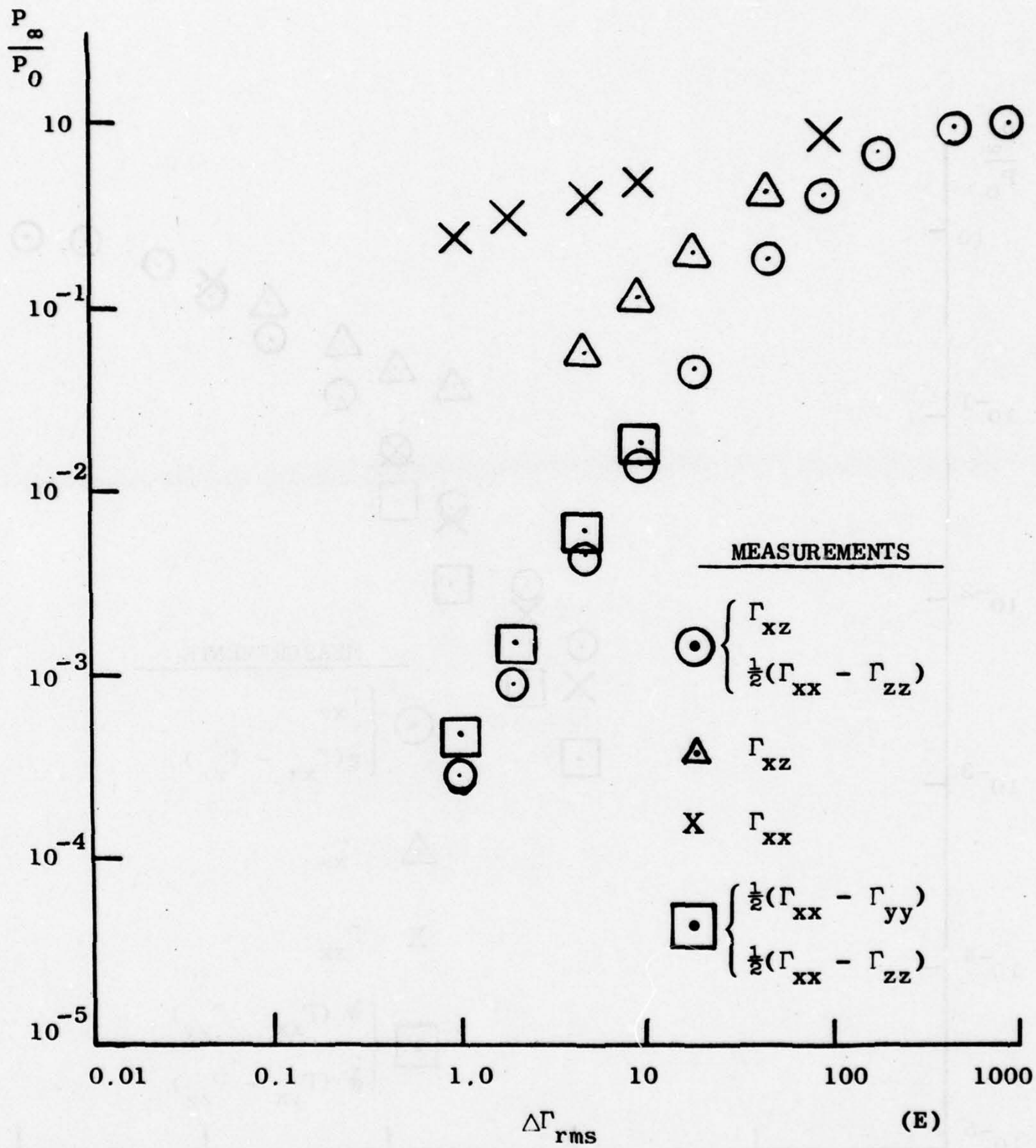


FIG. I-4 ESTIMATION ERROR COVARIANCE OF GRAVITY ANOMALY (g_z)

At this time, we do not understand fully the reason for Conclusion 2. A possible explanation is as follows.

For a one-dimensional gravity source, we have an identity, namely, $\Gamma_{yy} = - g/D$. Since $\Gamma_{xx} + \Gamma_{yy} + \Gamma_{zz}$ is always zero, we find that $\Gamma_{zz} = -\Gamma_{xx} + g/D$. Therefore, Γ_{zz} has not a weak but strong correlation with both g_x and g. Since $\frac{1}{2}(\Gamma_{xx} - \Gamma_{zz}) = \Gamma_{xx} - g/2D$, we can say that, qualitatively, the double measurement provides a better estimate of g than that from Γ_{xz} alone. On the contrary, it cannot provide better estimate of g_x than that from Γ_{xx} alone. However, since we have a good estimate of g, at least for relatively low measurement noise, the measurement $\frac{1}{2}(\Gamma_{xx} - \Gamma_{zz})$ gives a satisfactory estimate of Γ_{xx} and hence of g_x. For example, for 1 Eötvos measurement noise rms value with 10 sec average, the estimation error covariance of g/D is obtained to be on the order of 0.001 E. Assuming that the correlation time is on the order of 20 mins, we can say that the maximum power spectral density level is of the order 10^{-21} (sec^{-3}); hence, much smaller than that of the measurement error which is of the order 10^{-17} (sec^{-3}).

The computer program is shown in Appendix 2. The computer language used is not FORTRAN but APL (A Programming Language).

B-6 Wiener Filter Theory

If the filtered estimate $\hat{y}$ of some stationary process y, based on a noisy vector measurement $\overset{m\times 1}{z} + n$, where n is m-dimensional white noise, is given by:

$$y(s) = \overset{1\times m}{\psi^{T}}(s)[z(s) + n(s)] \tag{1.15}$$

where the transfer function vector $\psi(s)$ has only left half plane (LHP) poles, the mean-squared value of the estimation error, $\tilde{y} = \hat{y} - y$, is:

$$\sigma^2_{\tilde{y}} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Big\{\psi^{T}(-s)(\overset{m\times m}{\Phi_{zz}}(s) + \Phi_n)\psi(s) - \psi^{T}(-s)\Phi_{zy}(s) - \Phi_{zy}(-s)\psi(s) + \varphi_{yy}(s)\Big\}ds\,, \tag{1.16}$$

where $\varphi_{yy}(j\omega)$ is the spectral density of y, the Fourier transform of its ACF (autocorrelation function), i.e.,

$$\varphi_{yy}(j\omega) = \int_{-j\infty}^{j\omega} C_y(x)\cos\omega x dx\,,$$

($C_y(x)$ being the ACF of $y(x)$), and Φ_{zz}, Φ_{zy} are the transforms of the $\overset{m\times m}{\text{ACF}}$ of the vector z and of its cross-correlation with y.

The optimum $\psi(s)$ is determined by the requirements:

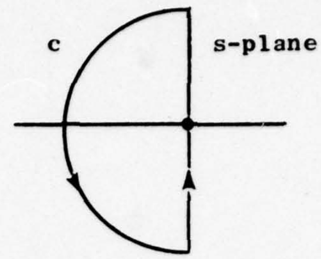
1) $\sigma^2_{\tilde{y}}$ in (1.15) must be finite;

2) $(\Phi_{zz}(s) + \Phi_n)\psi(s) - \Phi_{zy}(s)$ has no LHP poles. (1.17)

The minimum mean-squared estimation error is then

$$\sigma^2_{\tilde{y}} = \frac{1}{2\pi j}\int_{c}\{\varphi_{yy}(s) - \Phi_{zy}(-s)\psi(s)\}ds\,, \tag{1.18}$$

where c is any contour large enough to include all the LHP poles of the integrand in (1.17).



B-7 Asymptotic Results for Flight Measurement Accuracy

We will apply the method of the previous section, B-6, to the estimation of $g_z(= -g)$ from the single measurement Γ_{xz}, the spatial derivative of g_z in the direction of motion. The continuous analogue of the second component of equation (1.7) is:

$$g_z = \frac{\gamma D}{r^3} \otimes \rho \tag{1.19}$$

where ρ is linear density, a random process, and $\otimes$ denotes convolution. Now the Fourier transform of $\gamma D/r^3$ is

$$\int_{-\infty}^{\infty} \frac{\gamma D \cos \omega x}{(x^2 + D^2)^{\frac{3}{2}}} dx = \frac{2\gamma}{D} \Omega K_1(\Omega) , \tag{1.20}$$

where $\Omega = \omega D$, and K_1 is a modified Bessel function. Hence,

$$\varphi_{g_z g_z}(j\omega) = \frac{4\gamma^2}{D^2} \Omega^2 K_1^2(\Omega) \varphi_\rho \tag{1.21}$$

and $\varphi_{\Gamma_{xz},qz}$ and $\varphi_{\Gamma_{xz}\Gamma_{xz}}$ are obtained from (1.21) by multiplication by

by $-j\omega$ and ω^2. Φ_ρ is the earlier q

To proceed further, we need a rational approximation to $\Omega K_1(\Omega)$. A suitable approximation here is

$$\Omega K_1(\Omega) \approx \frac{22.173}{(2.252^2 + \Omega^2)^2} . \tag{1.22}$$

The related function $\Omega^2 K_1(\Omega)$ and its approximant are shown in Fig. I-5. ($\Omega^2 K_1(\Omega)$ is proportional to the transform: Γ_{zx}/ρ.)

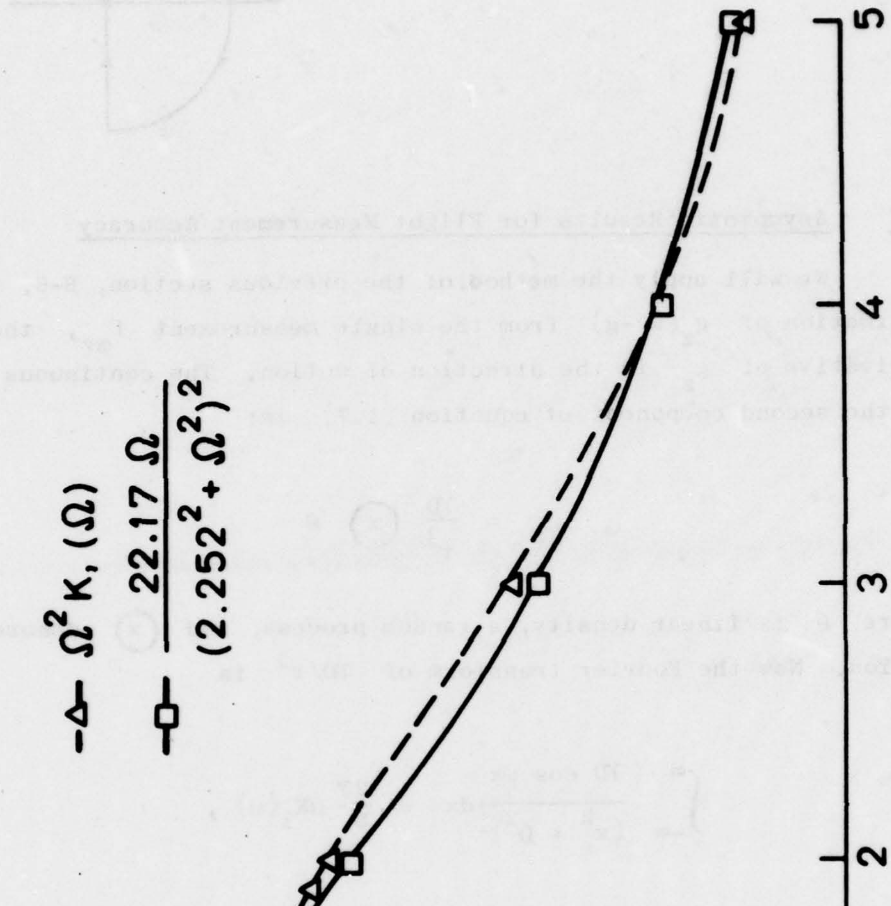


FIGURE I-5

Introducing $S = SD$, condition 2 of Eq. (1.17) becomes

$$\left[\frac{4B^2\gamma^2\varphi_\rho(-S^2)}{D^4(a^2 - S^2)^4} + \varphi_n\right]\psi(S) + \frac{4B^2\gamma^2 S}{D^3(a^2-S^2)^4} \tag{1.23}$$

has no LHP poles; where $B = 22.173$, $a = 2.252$.

The optimal transfer function $\psi(S)$ is thus given by

$$1 - \frac{S}{D}\psi(S) = \frac{(S + a)^4}{\prod_{j=i}^{4}(S + S_j)} \tag{1.24}$$

where $-S_j (j = 1, \ldots, 4)$ are the four LHP roots of the equation

$$D^4\varphi_n(a^2 - S^2)^4 = 4B^2\varphi_\rho S^2 . \tag{1.25}$$

Now, for <u>small</u> φ_n, Eq. (1.25) has a small LHP root:

$$S \cong -\frac{a^4D^2}{2\gamma B}\sqrt{\frac{\varphi_n}{\varphi_\rho}} \tag{1.26}$$

as well as three large LHP roots given by

$$S^6 - \frac{4B^2 q\varphi_\rho}{D^4\varphi_n} \cong 0 .$$

A low-frequency approximation to (1.24) is thus:

$$1 - s\psi(s) = 1 - \frac{S}{D}\psi(s) = \frac{a^4}{\frac{2B\gamma}{D^2\sqrt{\varphi_n/\varphi_\rho}}\left(S + \frac{a^4 D^2}{2B\gamma}\sqrt{\frac{\varphi_n}{\varphi_\rho}}\right)} = \frac{1/L}{s + \frac{1}{L}} \tag{1.27}$$

where $$L = \frac{2\gamma B}{a^4 D\sqrt{\varphi_n/\varphi_\rho}} .$$

The result (1.27) could have been obtained more directly by replacing $\Omega K_1(\Omega)$ by B/a^4 ($\cong 0.87$ instead of unity, which is really the correct limit as $\Omega \to 0$) and requiring that

$$\begin{cases} \frac{4B^2\gamma^2}{a^8 D^2} s[1 - s\psi(s)] + \varphi_n\psi(s) & \text{have no LHP poles} \\ 1 - s\psi(s) \to 0 \quad \text{as} \quad s \to \infty \end{cases} \tag{1.28}$$

The mean-squared estimation error is

$$\sigma^2_{g_z} = \frac{1}{2\pi j}\oint_C \frac{4B^2\gamma^2\varphi_\rho}{a^8 D^2}[1 - s\psi(s)]ds = \frac{4B^2\gamma^2\varphi_\rho}{a^8 D^2 L} \tag{1.29}$$

If, for example, measurement accuracy is 1 E every 10 sec at speed 50 m/s, $\varphi_n = 0.5\times10^{-18}$ km/sec^4, and, using D = 36 km, L = 8700 km, and $\sigma^2_{\tilde{g}_z}/\sigma^2_{g_z} = 1.1\times10^{-2}$. The straight line in Fig. I-4 shows the asymptotic behavior of this ratio according to (1.29). Note that L is "integration length" for the first-order integrator" $\psi(s) = 1/(s+ 1/L)$. The numerical value 8700 km is certainly excessive for some applications.

We turn next to the estimation of g_x from the single measurement Γ_{xx}. Since

$$g_x = \frac{\gamma x}{r^3} \otimes \rho ,$$

and $\gamma x/r^3$ has Fourier transform

$$\frac{2\gamma}{D}\frac{d}{d\Omega}\left[\Omega K_1(\Omega)\right] .$$

Low frequency approximation to $\varphi_{g_x g_x}$ is

$$\varphi_{g_x g_x} = \frac{4\gamma^2}{d^2}\left(\frac{-4B\Omega}{a^6}\right)^2 \varphi_\rho , \tag{1.30}$$

and $\varphi_{\Gamma_{xx}, g_x}$ and $\varphi_{\Gamma_{xx}\Gamma_{xx}}$ are obtained by multiplication by $-j\omega$ and ω^2.

The optimum filter transfer-function is thus obtainable from the requirements

$$\begin{cases} \dfrac{-64B^2\gamma^2\varphi_\rho}{a^{12}} s^3[1 - s\psi(s)] + \varphi_n\psi(s) & \text{has no LHP poles;} \\ s[1 - s\psi(s)] \to 0 \quad \text{as} \quad s \to \infty . \end{cases} \tag{1.31}$$

Hence

$$\psi(s) = \frac{s + \frac{2}{L}}{s^2 + \frac{2s}{L} + \frac{2}{L^2}} \tag{1.32}$$

where $(1/4)L^4 = \dfrac{64B^2\gamma^2\varphi_\rho}{a^{12}\varphi_n}$. The mean-squared estimation error is

$$\begin{aligned} \sigma_{g_x}^2 &= \frac{1}{2\pi j}\oint_c - \frac{64B^2\gamma^2 s^2\varphi_\rho}{\Omega^{12}}[1 - s\psi(s)]ds \\ &= \frac{64B^2\gamma^2\varphi_\rho}{a^{12}L^3} = \frac{4}{a^3}\sqrt{B\gamma}\ \varphi_n^{3/4}\ \varphi_\rho^{1/4} . \end{aligned} \tag{1.33}$$

If, again, measurement accuracy is 1 Eötvos every 10 sec (at speed 50 m/sec), we find:

$$L = 720 \text{ km}$$

and

$$\frac{\sigma^2_{\tilde{g}_x}}{\sigma^2_{g_x}} = 0.26 \times 10^{-2} .$$

The straight line in Fig. I-3 shows the asymptotic behavior of this ratio, according to (1.33).

Chapter II

TWO-DIMENSIONAL GRAVITY SOURCE TREATMENT

A. THE HELLER MODEL

This model assumes several independent layers below the earth surface, of "white-noise" potential variations U_{w_i} $(i = 1, \dots 5)$. The external gravity potential fluctuation U at radius r from the earth's center is expressible by Poisson's integral formula

$$U(r,\theta,\varphi) = \sum_i \frac{R_i(r^2 - R_i^2)}{4\pi} \int \frac{U_{w_i}}{(r^2 + R_i^2 - 2rR\cos\psi)^{3/2}} \, d\Omega' . \qquad (1)$$

where R_i is the radius of the i-th layer, ψ denotes the angle between directions (θ, φ) and (θ', φ'), and $d\Omega' = \sin\theta' \, d\theta' \, d\varphi'$.

For the description of fluctuations with wavelengths substantially less than earth radius a 'flat earth' approximation to (1) is sufficient:

$$U(x,y,z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \sum_i \frac{(z + D_i) U_{w_i}(x',y')}{[(x-x')^2 + (y-y')^2 + (z+D_i)^2]^{\frac{3}{2}}} \, dx'dy', \qquad (2)$$

where z is altitude above earth surface, and D_i in the depth of the i-th layer.

Note that formula (2) corresponds <u>not</u> to two-dimensional white-noise mass density variations at depths D_i but rather to layers of random mass density "doublets":

$$\cdots \pm \pm \pm \mp \pm \mp \mp \pm \pm \cdots$$

Heller proposes five separate layers but only the three shallowest layers contribute appreciably to the short wavelength fluctuations.

Table 1 gives the depths D_i and the spectral densities φ_i for the potential variations U_{w_i}

Table 1

i	D_i (km)	φ_i (km^6/sec^4)
1	16.3	7.13×10^{-8}
2	92.5	1.07×10^{-4}
3	390.5	1.16×10^{-2}

The two-dimensional Fourier transform of relation (2) takes a simple form

$$U[\omega_x, \omega_y; z] = \sum_i e^{-z_i \omega} U_{w_i}(\omega_x, \omega_y) \tag{3}$$

where

$$\begin{cases} z_i = z + D_i \\ \omega = \sqrt{\omega_x^2 + \omega_y^2} \end{cases}$$

and $U_{w_i}[\omega_x, \omega_y]$ denotes the transform of $U_{w_i}(x, y)$:

$$U_{w_i}[\omega_x, \omega_y] = \iint_{-\infty}^{\infty} e^{-j(\omega_x x + \omega_y y)} U_{w_i}(x, y)\, dxdy \quad .$$

The two-dimensional transforms of gravity fluctuations g_x, g_y, g_z are obtained by multiplying (3) by $j\omega_x$, $j\omega_y$, and $-\omega$ respectively. The mean-squared fluctuations in g_x, g_y, g_z are thus obtainable:

$$(\sigma^2_{g_x}, \sigma^2_{g_y}, \sigma^2_{g_z}) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} (\omega_x^2, \omega_y^2, \omega_z^2) \sum_i \varphi_i e^{-2z_i\omega} d\omega_x d\omega_y .$$

This leads easily (substituting $\omega_x = \omega \cos \alpha$, $\omega_y = \omega \sin \alpha$) to

$$\sigma^2_{g_x} = \sigma^2_{g_y} = \frac{1}{2}\sigma^2_{g_z} = \sum_i \frac{\varphi_i}{4\pi} \int_0^{\infty} e^{-2z_i\omega} \omega^3 d\omega = \frac{3}{2\pi} \sum_i \frac{\varphi_i}{(2z_i)^4} \tag{4}$$

The two-dimensional Fourier transform of Γ_{yz}, Γ_{zz}, etc., are obtained from (3) by multiplication by $-j\omega_y\omega$, ω^2, etc. The two-dimensional spectral-density matrix relating g_x, g_y, g_z, Γ_{yz}, Γ_{zz} is thus

$$\sum_i \varphi_i e^{-2z_i\omega} \begin{pmatrix} \omega_x^2 & \omega_x\omega_y & j\omega_x\omega & -\omega_x\omega_y\omega & -j\omega_x\omega^2 \\ \omega_x\omega_y & \omega_y^2 & j\omega_y\omega & -\omega_y^2\omega & -j\omega_y\omega^2 \\ -j\omega_x\omega & -j\omega_y\omega & \omega^2 & -j\omega_y\omega^2 & -\omega^3 \\ -\omega_x\omega_y\omega & -\omega_y^2\omega & j\omega_y\omega^2 & \omega_y^2\omega^2 & j\omega_y\omega^3 \\ j\omega_x\omega^2 & j\omega_y\omega^2 & -\omega^3 & -j\omega_y\omega^3 & \omega^4 \end{pmatrix} \tag{5}$$

We need, of course, the one-dimensional spectral-density matrix along the line $y = 0$. This is obtained by the operation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (\;)\, d\omega_y$$

We thus obtain, for each layer i, and for states $g_x/2z_i$, $g_y/2z_i$, $g_z/2z_i$, Γ_{yz}, Γ_{zz}:

$$\frac{\varphi_i}{\pi(2z_i)^5}\begin{pmatrix} \Omega^3 K_1(\Omega) & 0 & -j\Omega^3 K_1'(\Omega) & 0 & -j\Omega^4 K_1''(\Omega) \\ 0 & \Omega^3[K_1''(\Omega)-K_1(\Omega)] & 0 & \Omega^4[K_1'''(\Omega)-K_1(\Omega)] & 0 \\ j\Omega^3 K_1'(\Omega) & 0 & \Omega^3 K''(\Omega) & 0 & \Omega^4 K_1'''(\Omega) \\ 0 & \Omega^4[K_1'''(\Omega)-K_1'(\Omega)] & 0 & \Omega^5[K_1''''(\Omega)-K_1''(\Omega)] & 0 \\ j\Omega^4 K_1''(\Omega) & 0 & \Omega^4 K_1'''(\Omega) & 0 & \Omega^5 K_1''''(\Omega) \end{pmatrix}$$

where

$$K_1(\Omega) = \int_0^{\infty} e^{-\Omega\cosh u} \cosh u \; du$$

a modified Bessel function.

The gravity-gradient components Γ_{xx}, Γ_{xy}, Γ_{xz}, which are the spatial derivatives g_x, g_y, g_z is the direction of motion, are easily included by multiplication by $\pm j\omega_x$. The component Γ_{yy} is obtainable from Γ_{xx} and Γ_{zz} by Laplace's equation.

The zero correlation between g_y, Γ_{xy}, and the remaining states implies, of course, that the optimal estimate of g_y can involve only the measurements Γ_{xy}, Γ_{yz}, while the optimal estimates of g_x and g_z can involve only the measurements Γ_{xx}, Γ_{xz}, Γ_{zz} (or Γ_{yy}).

B. ASYMPTOTIC RESULTS FOR VERY ACCURATE MEASUREMENTS

In accordance with the results obtained with the one-dimensional density fluctuation model, we anticipate that, in the case of very accurate measurements, a low-frequency approximation to the spectral-density matrix will suffice. The low-frequency form of (6) is:

$$\frac{\varphi_i}{\pi(2z_i)^5}\begin{pmatrix} \Omega^2 & 0 & j\Omega & 0 & -2j\Omega \\ 0 & 2 & 0 & -6 & 0 \\ -j\Omega & 0 & 2 & 0 & -6 \\ 0 & -6 & 0 & 24 & 0 \\ 2j\Omega & 0 & -6 & 0 & 24 \end{pmatrix} \qquad (7)$$

Hence, for several layers, the low-frequency optical-density matrices for g_y, Γ_{yz} and for g_x, g_z, Γ_{zz} are, respectivly:

$$\begin{pmatrix} 2k_3 & -6k_4 \\ -6k_4 & 24k_5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -k_1 s^2 & k_2 s & -2k_3 s \\ -k_2 s & 2k_3 & -6k_4 \\ +2k_3 s & -6k_4 & 24k_5 \end{pmatrix} \qquad (8)$$

where $k_j = 1/\pi \sum_i [\varphi/(2z_i)^j]$ and s denotes $j\omega_x$.

Example (1): Estimation of g_y from Γ_{xy} (or of g_x from Γ_{xz})

$$\text{If} \begin{cases} \hat{g}_y = \psi(s)\,(\Gamma_{xy} + n) \\ \tilde{g}_y = \hat{g}_y - g_y \end{cases} .$$

Then, according to Eq. (1.16) and Eq. (8):

$$\sigma^2_{\tilde{g}_y} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Big\{\psi(-s)(-2k_3 s^2 + \varphi_n)\psi(s) + \psi(-s)(2k_3 s) + (-2k_3 s)\psi(s) + 2k_3\Big\} ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \{2k_3[1 + s\psi(-s)][1 - s\psi(s)] + \varphi_n \psi(-s)\psi(s)\} ds$$

which is minimized by choosing $\psi(s)$ so that

$$\begin{cases} 1 - s\psi(s) \to 0 \quad \text{as} \quad s \to \infty \\ (\varphi_n - 2k_3 s^2)\psi(s) \quad \text{has no L.H P poles.} \end{cases}$$

Hence $\psi(s) = 1/[s+(1/L)]$, a "first-order integrator" with "integration length" $L = \sqrt{2k_3/\varphi_n}$.

The mean-squared estimation error is

$$\sigma^2_{\tilde{g}_y} = \frac{1}{2\pi j}\oint_c 2k_3[1 - s\psi(s)]ds = \sqrt{2k_3\varphi_n} .$$

This result is entirely similar to that obtained earlier with the one-dimensional density fluctuation model.

If we suppose again that independent measurements of Γ_{xy} (or Γ_{xz}) with an accuracy of 1 Eötvos are obtained every 10 sec at a velocity of 50 m/sec, i.e., every 0.5 km, then $\varphi_n = 0.5 \times 10^{-18}$ km/sec^4. If the altitude z is zero, we find, with the aid of Table 1, $\sigma_{\tilde{g}_z} = \sigma_{\tilde{g}_z} = 6.4 \times 10^{-6}$ g, not too different from the $\sigma_{\tilde{g}_z}$ obtained in Chap I. The integration length L = 7900 km which may again be excessive for some applications! The fluctuations themselves have rms values:

$$\sigma_{g_z} = \sqrt{2}\ \sigma_{g_y} = \sqrt{3k_4} = 4.6 \times 10^{-5} g$$

(The three shallowest layers of the Heller model give a somewhat lower value than assumed earlier.)

Example 2: Estimation of g_y from Γ_{xy} and Γ_{yz} (or of g_z from Γ_{xz} and Γ_{zz}.)

If $\hat{g}_y = \psi_1(s)(\Gamma_{xy} + n_1) + \psi_2(s)(\Gamma_{yz} + n_2)$,

$$\sigma^2_{\tilde{g}_y} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left\{ [\psi_1(-s),\ \psi_2(-s)] \begin{pmatrix} -2k_3 s^2 + \varphi_n & 6k_4 s \\ -6k_4 s & 24k_5 + \varphi_n \end{pmatrix} \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} \right.$$

$$\left. + [\psi_1(-s), \psi_2(-s)] \begin{pmatrix} 2k_3 s \\ 6k_4 \end{pmatrix} + (-2k_3 s,\ 6k_4) \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} + 2k_3 \right\} ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left\{ 2k_3[1+s\psi_1(-s)][1-s\psi_1(0)] + \varphi_n \psi_1(-s)\psi_1(s) \right.$$

$$+ 6k_4\psi_2(-s)[1 - s\psi_1(s)]$$

$$\left. + 6k_4[1+s\psi_1(-s)]\psi_2(s) + (24k_5+\varphi_n)\psi_2(-s)\psi_2(s) \right\} ds .$$

This is minimized by choosing $\psi_1(s)$, $\psi_2(s)$ so that

$$\begin{cases} 1 - s\psi_1(s) \to 0 & \text{as } s \to \infty \\ \psi_2(s) \to 0 & \text{as } s \to \infty \end{cases}$$

$$\begin{pmatrix} -2k_3 s^2 + \varphi_n & 6k_4 s \\ -6k_4 s & 24k_5 + \varphi_n \end{pmatrix} \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} \quad \text{has no LHP poles.}$$

The asymptotic result for small φ_n is:

$$\psi_1(s) = \frac{1}{s + (\frac{1}{L})}, \quad \psi_2(s) = \frac{-\lambda}{s + (\frac{1}{L})},$$

where $\lambda = k_4/4k_5L$, and where the integration length L is now:

$$L = \sqrt{\frac{2k_3}{\varphi_n}\left(1 - \frac{3k_4^2}{4k_3k_5}\right)} .$$

The resulting mean-squared estimation error is:

$$\sigma_{\tilde{g}_y} = \frac{1}{2\pi j} \oint_c \{2k_3[1 - s\psi_1(s)] + 6k_4\psi_2(s)]\}ds$$

$$= \frac{2k_3}{L} - 6\lambda k_4 = \sqrt{2k_3\varphi_n\left(1 - \frac{3k_4^2}{4k_3k_5}\right)} .$$

In our numerical example, the additional information provided by the additional measurement is not substantial; $\sigma_{\tilde{g}_y}$ is reduced by $7\frac{1}{2}$ percent, and L by 15 percent.

<u>Example 3</u>: Estimation of g_x from Γ_{xx}

If $\quad g_x = \psi(s)(\Gamma_{xx} + n)$,

$$\sigma^2_{\tilde{g}_x} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \{\psi(-s)(k_1s^4+\varphi_n)\psi(s) + k_1s^3\psi(s)-\psi(-s)k_1s^3-k_1s^2\}ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \{(-k_1s^2)[1+s\psi(-s)][1-s\psi(s)] + \varphi_n\psi(-s)\psi(s)\}ds ,$$

which is minimized by choosing $\psi(s)$ so that

$$\begin{cases} s[1 - s\psi(s)] \to 0 & \text{as} \quad s \to \infty \\ (k_1 s^4 + \varphi_n)\psi(s) & \text{has no LHP poles} . \end{cases}$$

Hence,

$$\psi(s) = \frac{s + \frac{2}{L}}{s^2 + \frac{2s}{L} + \frac{2}{L^2}} ,$$

a "second-order integrator" with integration length $L = (4k_1/\varphi_n)^{1/4}$.

The mean-squared estimation error is

$$\sigma^2_{\tilde{g}_x} = \frac{1}{2\pi j} \oint_c (-k_1 s)[1 - s\psi(s)]ds = \frac{4k_1}{L^3} = (4k_1 \varphi_n^3)^{1/4} .$$

This, again, is similar to the result obtained with the one-dimensional gravity fluctuation model.

In our numerical example, $\sigma_{\tilde{g}_x} = 3. \times 10^{-6} g$ and $L = 2500$ km.

<u>Example 4</u>: Estimation of g_x from Γ_{xx} <u>and</u> Γ_{xz} .

If $\hat{g}_x = \psi_1(s)(\Gamma_{xx} + n_1) + \psi_2(s)\Gamma_{xz}$,

$$\sigma^2_{\tilde{g}_x} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Bigg\{ [\psi_1(-s), \psi_2(-s)] \begin{pmatrix} k_1 s^4 + \varphi_n & -k_2 s^3 \\ k_2 s^3 & -2k_3 s^2 + \varphi_n \end{pmatrix} \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix}$$

$$+ [\psi_1(s), \psi_2(-s)] \begin{pmatrix} -k_1 s^3 \\ -k_2 s^2 \end{pmatrix} + (k_1 s^3, -k_2 s) \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} - k_1 s^2 \Bigg\} ds$$

= (cont'd next page)

$$= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Big\{ -k_1 s^2 [1+s\psi_1(-s)][1-s\psi(s)] + \varphi_n \psi_1(-s)\psi_1(s) - k_2 s^2 \psi_2(-s)[1-s\psi_1(s)] - k_2 s^2 [1+s\psi_1(-s)]\psi_2(s) + \psi_2(-s)(-2k_3 s^2 + \varphi_n)\psi_2(s) \Big\} ds ,$$

which is minimized by choosing $\psi_1(s)$, $\psi_2(s)$ so that

$$\begin{cases} s[1 - s\psi_1(s)] \to 0 & \text{as} \quad s \to \infty \\ s\psi_2(s) \to 0 & \text{as} \quad s \to \infty \\ \begin{pmatrix} k_1 s^4 + \varphi_n & -k_2 s^3 \\ k_2 s^3 & -2k_3 s^2 + \varphi_n \end{pmatrix} \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} & \text{has no LHP poles} . \end{cases}$$

The asymptotic solution, for small φ_n, is:

$$\psi_1(s) = \frac{s + \frac{2}{L}}{s^2 + \frac{2s}{L} + \frac{2}{L^2}} , \qquad \psi_2(s) = \frac{-\lambda/L}{s^2 + \frac{2s}{L} + \frac{2}{L^2}}$$

where

$$\lambda = \frac{k_2}{k_3 L} \qquad \text{and} \qquad L = \left[\frac{4k_1}{\varphi_1} \left(1 - \frac{k_2^2}{2k_1 k_3} \right) \right]^{1/4} .$$

The mean-squared estimation error is

$$\sigma^2_{\tilde{g}_x} = \frac{1}{2\pi j} \oint_c \{-k_1 s^2 [1-s\psi_1(s)] - k_2 s^2 \psi_2(s)\} ds$$

$$= \frac{4k_1}{L^3} \left(1 - \frac{\lambda k_2 L}{2k_1} \right) = \left[4k_1 \varphi_n^3 \left(1 - \frac{k_2^2}{2k_1 k_3} \right) \right]^{1/4} .$$

Again the improvement due to the additional information, in this case Γ_{xz}, is minor. In our numerical example L is reduced by 10 percent and σ_{g_x} by only 5 percent.

A similar analysis applies to the inclusion of a 3rd measurement, Γ_{zz}, in the estimation of g_x. The improvement turns out to be equally minor.

However, the inclusion of the measurement, Γ_{xx}, in the estimation of g_z from Γ_{xz} does not lead to a simple "slow" filter. The results from the one-dimensional density fluctuation model suggest that there may be a substantial improvement over the first order integration of Γ_{xz}, although this may have been related to the identity $\Gamma_{yy} = -\frac{1}{5} g_z$ which applies only to the one-dimensional fluctuation model.

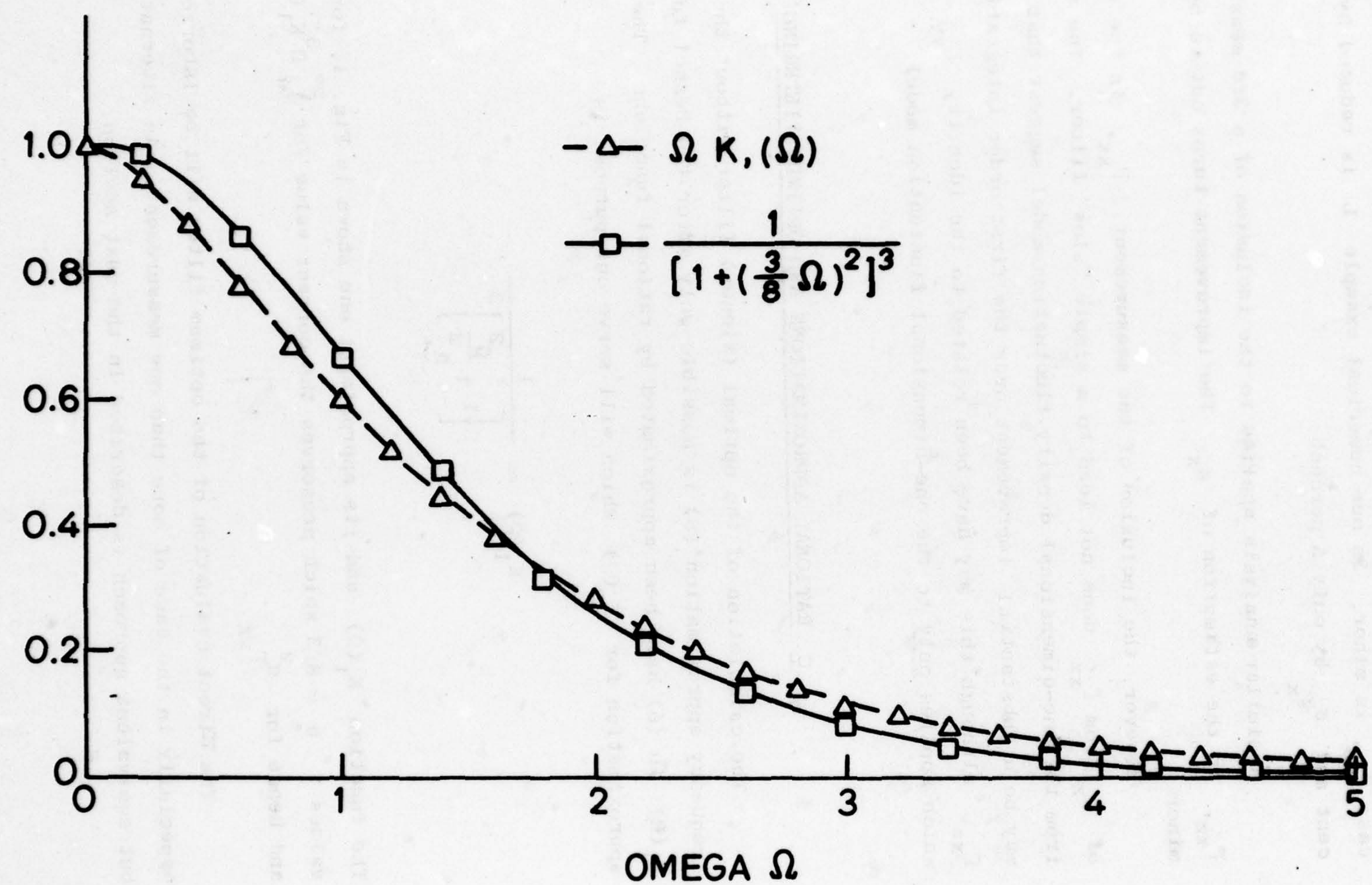
C. RATIONAL APPROXIMATIONS AND OPTIMUM FILTERING

The calculation of the optimal (Wiener) filter without the low-frequency approximation (8) is possible only after the Bessel functions $K_1(\Omega)$ in (6) have been approximated by rational functions. The simplest approximation for $K_1(\Omega)$ which will serve our purpose is

$$K_1(\Omega) \approx \frac{1}{\Omega\left[1 + \frac{\Omega^2}{b^2}\right]^3} . \qquad (9)$$

The function $K_1(\Omega)$ and its approximant one shown in Fig. 1, for the values b = 8/3 which preserves the correct value for $\int_0^\infty \Omega^5 K_1(\Omega) d\Omega$ and hence for $\sigma^2_{\Gamma_{xx}}$.

The direct evaluation of the optimum filter will be laborious, especially in the case of more than one measurement. An alternative but equivalent approach is described in the next section.



$\Omega K_1(\Omega)$
$\frac{1}{[1+(\frac{3}{8}\Omega)^2]^3}$
1.0
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
OMEGA Ω

FIGURE II-1

D. FINITE STATE MODEL

We will discuss how to construct a model in which the components of $\vec{g}$ and of the gravity-gradient tensor Γ form part of a finite state which obeys a linear evolution in response to a finite white-noise input. The optimal estimates and their accuracy will then be obtainable from the steady-state solution of a standard matrix-Riccati equation. For <u>each</u> layer (i = 1, 2, 3) we build a model for g_y and Γ_{yz} as follows:

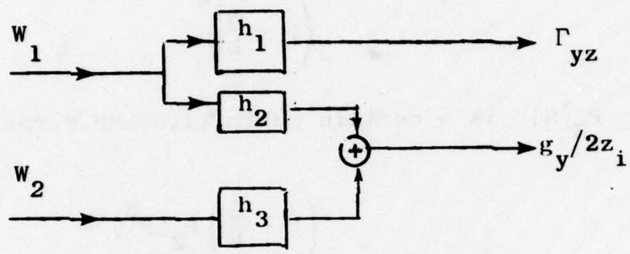


FIGURE II-2

$$\Gamma_{yz} = h_1(S)W_1$$

$$g_y/2z_i = h_2(S)W_1 + h_3(S)W_2$$

where S denotes $j\Omega$, and W_1, W_2 are independent white-noise inputs with spectral density $\varphi_i/\pi(2z_i)^5$. The transfer functions $h_i(S)$ must lead to the rational approximation, obtained with (9), to the spectral-density matrix:

$$
\begin{array}{ll}
\Gamma_{yz}: & \multirow{2}{*}{$\begin{pmatrix} \Omega^5[K_1''''(\Omega)-K_1''(\Omega)] & \Omega^4[K_1'''(\Omega) - K_1'(\Omega)] \\ \Omega^4\, K_1'''(\Omega) -K_1'(\Omega)] & \Omega^3[K''(\Omega) -K(\Omega)] \end{pmatrix}$} \\
g_y/\partial z_i: &
\end{array}
$$

This leads to:

(a)
$$h_1(s) = \frac{P_3(-s)}{\left(1 + \frac{S}{b}\right)^6}$$

where $P_3(S)$ is a certain cubic with LHP zeros;

(b)
$$h_2(S) = \frac{\left(1 - \frac{S}{b}\right) P_2(S^2)}{\left(1 + \frac{S}{b}\right)^5 P_3(S)}$$

where P_2 is a certain quadratic polynomial;

(c)
$$h_3(S\ ) = \frac{P_4(\pm\ S)}{\left(1 + \frac{S}{b}\right)^5 P_3(S)}$$

where P_4 is a certain quartic.

The model for g_x, g_z, Γ_{zz} is more elaborate, as shown in Fig. II-3.

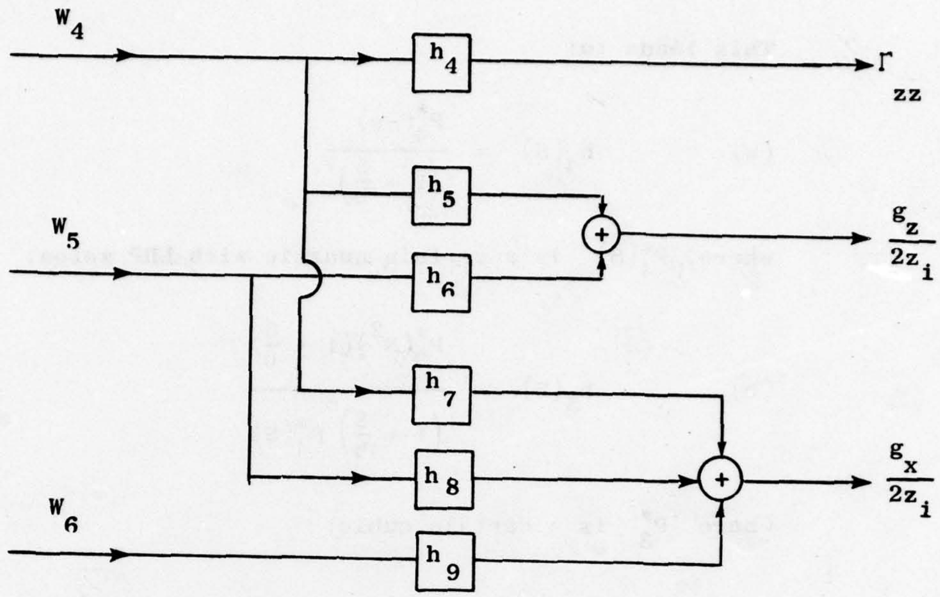


FIGURE II-3

$$
\left\{
\begin{aligned}
\Gamma_{zz} &= h_4(S)W_4 \\
g_z/2z_i &= h_5(S)W_4 + h_6(S)W_5 \\
g_x/2z_i &= h_7(S)W_4 + h_8(S)W_5 + h_9(S)W_6
\end{aligned}
\right.
$$

where again W_4, W_5, W_6 are independent white-noise inputs with spectral-density $\varphi_i/\pi(2z_i)5$, and the transfer-functions $h_j(S)(j = 4 - 9)$ lead to the rational approximation, obtained with (2.9), to the spectral-density matrix:

$$
\begin{array}{l}
\Gamma_{zz}: \\
g_z/2z_i: \\
g_x/2z_i:
\end{array}
\begin{pmatrix}
\Omega^5 K_1'''(\Omega) & \Omega^4 K_1'''(\Omega) & j\Omega^4 K_1''(\Omega) \\
\Omega^4 K_1'''(\Omega) & \Omega^3 K_1''(\Omega) & j\Omega^3 K_1'(\Omega) \\
-j\Omega^4 K''(\Omega) & -j\Omega^3 K_1'(\Omega) & \Omega^3 K_1(\Omega)
\end{pmatrix}.
$$

This leads to:

(a) $$h_4(S) = \frac{P_4^*(-s)}{\left(1 + \frac{S}{b}\right)^7}$$

where $P_4^*(S)$ is a certain quartic with LHP zeros;

(b) $$h_5(S) = \frac{P_3^*(S^2)(1 - \frac{S}{b})}{\left(1 + \frac{S}{b}\right)^6 P_4^*(S)}$$

where P_3^* is a certain cubic;

(c) $$h_6(S) = \frac{P_6(-S)}{\left(1 + \frac{S}{b}\right)^5 P_4^*(S)}$$

where $P_6(S)$ is a certain sextic with LHP zeros;

(d) $$h_7(S) = \frac{S\, P_2^*(S^2)(1 - \frac{S}{b})^2}{\left(1 + \frac{S}{b}\right)^5 P_4^*(S)}$$

where P_2^* is a certain quadratic;

(e) $$h_8(S) = \frac{S\, P_5(S^2)(1 - \frac{S}{b})}{\left(1 + \frac{S}{b}\right)^4 P_4^*(S) P_6(S)}$$

where P_5 is a certain quintic;

(f) $$h_9(S) = \frac{S\, P_{10}(\pm S)}{\left(1 + \frac{S}{b}\right)^3 P_4^*(S) P_6(S)}$$

where P_{10} is a certain 10-th degree polynomial.

The total effect of all three layers $(i = 1, 2, 3)$ is obtained by adding in parallel three appropriate versions of Figures II-2 or II-3.

It is planned to carry through this finite-state approximation and the resulting optimal filters, in order to obtain an (approximate) check on the range of validity of the asymptotic filters obtained for high measurement accuracy. It is hoped also to investigate the asymptotic form, if any, of the estimate of g_z when Γ_{xx} and Γ_{zx} are both measured, and to see if the improvement obtained with Γ_{xx} in the case of a one-dimensional density fluctuation model carries over to the more realistic model.

Appendix 1

CHANDRASEKHAR ALGORITHM

According to Kailath [Ref. 1], the Chandrasekhar type algorithm is described as follows.

Suppose we have a process with a known state-space model of the form

$$\left.\begin{aligned} y_i &= Hx_i + V_i \\ x_{i+1} &= Fx_i + GU_i \end{aligned}\right\} \quad i \geq 0$$

where

$$EU_iU_i' = Q\delta_{ij}, \qquad EV_iV_j' = R\delta_{ij}$$

$$EU_iV_j' = 0$$

$$Ex_0x_0' = \Pi_0 \qquad EU_ix_0' \equiv 0 \equiv EV_ix_0'.$$

Assume that $\{F, G, H, Q, R\}$ are constant matrices with dimensions $n \times n$, $n \times m$, $p \times n$, $m \times m$, $p \times p$.

Let

$$D = F\Pi_0F' + GQG' - F\Pi_0H'(R+H\Pi_0H')^{-1}H\Pi_0F' - \Pi_0$$

and assume that we can represent it (nonuniquely) as

$$D = L_0M_0L_0'$$

where L_0 and M_0 are $n \times \alpha$ and $\alpha \times \alpha$ matrices, $\alpha = \text{rank } D$, and

$$M_0 = \text{diag}\,\{1, 1, \ldots, 1, -1, -1, \ldots, -1\}$$

with as many $\pm$ 1's as D has $\pm$ eigenvalues. Then the quantities $\{K_i, R_i^{\epsilon}\}$ appearing in the estimator formula

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i-1} + K_i(R_i^{\epsilon})^{-1}(y_i - H\hat{x}_{i|i-1})$$

can be computed via the equations

$$R_{i+1}^{\epsilon} = R_i^{\epsilon} - HL_i(R_i^r)^{-1}L_i'H'$$

$$R_{i+1}^r = R_i^r - L_i'H'(R_i^{\epsilon})^{-1}HL_i$$

$$K_{i+1} = K_i - FL_i(R_i^r)^{-1}L_i'H'$$

$$L_{i+1} = FL_i - K_i(R_i^{\epsilon})^{-1}HL_i$$

where the initial values are

$$R_0^{\epsilon} = R + H\Pi_0H', \quad K_0 = F\Pi_0H', \quad R_0^r = -M_0^{-1}.$$

The number of computations required to go from index i to index $i+1$ can be seen to be $\mathcal{O}(n^2(p+\alpha))$ as compared to $\mathcal{O}(n^3)$ if the Riccati equation is used. For special initial conditions, matrices D, L_0, and M_0 are given as follows:

(a) $\Pi_0 = 0$ (perfect a priori knowledge of the states);

$$D = GQG', \quad L_0 = GQ^{\frac{1}{2}}, \text{ and } M_0 = I.$$

(b) $\Pi_0 = \bar{\Pi}$ (stationary process)

$$F\bar{\Pi}F' + GQG' = \bar{\Pi}$$

$$D = -\bar{\Pi}H'(R + H\bar{\Pi}H')^{-1}H\bar{\Pi}$$

$$L_0 = \bar{\Pi}H'(R + H\bar{\Pi}H')^{-T/2}$$

$$M_0 = -I.$$

If the error covariance matrix P_{i+1} is desired, it may be computed as

$$P_{i+1} = \Pi_0 - \sum_{k=0}^{i} L_K (R_k^r)^{-1} L_k' .$$

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$$P_{i+1} = \Pi_0 - \sum_{k=0}^{i} L_K (R_k^r)^{-1} L_k' .$$